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On Internal Fracture of Solids *#

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Three relevant analytical and experimental considerations on the initiation and propagation of internal fracture in a solid are carried out.

(a) The initiation of internal fracture in the vicinity of a point as a result of converging tensile pulses is analyzed. A reasonable estimate of the state of stress near one of the foci of a prolate spheroid is obtained.

(b) Experimental success in reflecting and focusing sharp pulses and thus fracturing the neighborhood of a focus in a prolate spheroid is observed. The size of the internal fracture ranging from a pinpoint to a volume having more than 3 mm in diameter has been obtained.

(c) The propagation of a single crack in a solid is then analyzed. Viscoelastic behavior of the medium is considered and the time-dependent fracture information is given.

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[redacted]

operator. Introducing [redacted]

$$\underset{\sim}{U} = \nabla \Phi + \nabla \times \underset{\sim}{\Psi} \qquad (2)$$

into (1), the following wave equations will be obtained.

$$\frac{\partial^2 \Phi}{\partial t^2} - C_1^2 \nabla^2 \Phi = 0 \qquad (3)$$

$$\frac{\partial^2 \overline{\Psi}}{\partial t^2} - C_2^2 \nabla^2 \underset{\sim}{\Psi} = 0 \qquad (4)$$

where $C_1^2 = \frac{\lambda + 2\mu}{m}$; $C_2^2 = \frac{\mu}{m}$

In spheroidal coordinates ξ , η , ϕ if a , b are the lengths of the major and minor axes respectively and $2f$, the interfocal distance of the prolate spheroid, then $a^2 - b^2 = f^2$ and [1]

$$\nabla^2 F = \frac{1}{f^2(\xi^2 - \eta^2)} \left[\frac{\partial}{\partial \xi}(\xi^2 - 1)\frac{\partial F}{\partial \xi} + \frac{\partial}{\partial \eta}(1 - \eta^2)\frac{\partial F}{\partial \eta} + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2 F}{\partial \phi^2} \right] \qquad (5)$$

In the case of axial symmetry (5) reduces to

$$\nabla^2 F = \frac{1}{f^2(\xi^2 - \eta^2)} \left[\frac{\partial}{\partial \xi}(\xi^2 - 1)\frac{\partial F}{\partial \xi} + \frac{\partial}{\partial \eta}(1 - \eta^2)\frac{\partial F}{\partial \eta} \right] \qquad (6)$$

and (4) reduces to

$$\frac{\partial^2 \overline{\Psi}}{\partial t^2} - C_2^2 \nabla^2 \overline{\Psi} = 0 \qquad (7)$$

because $\underset{\sim}{\Psi} = \Psi \underset{\sim}{e}_\phi$, where $\underset{\sim}{e}_\phi$ is the unit vector in the direction of increasing ϕ .

The boundary conditions are such that over the spheroidal surface both the normal and tangential stress components vanish. These conditions can be expressed in terms of the functions Φ and Ψ . Let $\frac{\lambda + 2\mu}{\lambda} \equiv K$, then for vanishing normal stresses

$$\frac{\mathcal{K} b}{f(1-\eta^2)^{\frac{1}{2}}} \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{f(1-\eta^2)^{\frac{1}{2}}}{b} \frac{\partial^2 \Phi}{\partial \eta^2} + (\mathcal{K}-1) \frac{\partial^2 \Phi}{\partial \xi \partial \eta}$$

$$+ \frac{a^2+b^2+\mathcal{K} f^2-(\mathcal{K}+1) f^2 \eta^2}{(a^2-f^2\eta^2)\, b\, (1-\eta^2)^{\frac{1}{2}}} \left(a \frac{\partial \Phi}{\partial \xi} - f \eta \frac{\partial \Phi}{\partial \eta} \right)$$

$$- \frac{(\mathcal{K}-1) f a}{a^2 - f^2 \eta^2} \frac{\partial \Psi}{\partial \eta} - \frac{(\mathcal{K}-1) b^2}{a^2-f^2\eta^2} \frac{\partial \Psi}{\partial \xi} + \frac{(\mathcal{K}-1) a f \eta}{b^2 (1-\eta^2)} \Psi = 0 \tag{8}$$

and for vanishing shear stresses

$$2 \frac{\partial^2 \Phi}{\partial \xi \partial \eta} + \frac{f(1-\eta^2)^{\frac{1}{2}}}{b} \frac{\partial^2 \Psi}{\partial \eta^2} - \frac{b}{f(1-\eta^2)^{\frac{1}{2}}} \frac{\partial^2 \Psi}{\partial \xi^2}$$

$$+ 2 \frac{f^2 \eta}{a^2-f^2\eta^2} \frac{\partial \Phi}{\partial \xi} - 2 \frac{f a}{a^2-f^2\eta^2} \frac{\partial \Phi}{\partial \eta} - 2 \frac{f b \eta}{(a^2-f^2\eta^2)(1-\eta^2)^{\frac{1}{2}}} \frac{\partial \Psi}{\partial \eta}$$

$$+ 2 \frac{f^2 a (1-\eta^2)^{\frac{1}{2}}}{b(a^2-f^2\eta^2)} \frac{\partial \Psi}{\partial \xi} + \frac{f}{b(1-\eta^2)^{\frac{1}{2}}} \left(2 + \frac{a^2}{b^2} - \frac{\eta^2}{1-\eta^2} \right) \Psi = 0 \tag{9}$$

These equations (8) and (9) must be satisfied on the surface of the spheroid $\xi = \frac{a}{f}$

In the neighborhood of the focus where detonation occurs the material is subjected to very large compressive stresses. As a result the equations of linear elasticity may not be applied. This situation can be avoided by choosing a sphere of radius ϵ

such that outside this sphere the compressive stresses are small enough to justify the use of the linear elasticity equations. Consequently, the inner boundary conditions over the spherical surface $\xi + \eta = \frac{\epsilon}{f}$ are describable by the following two additional equations. For normal stresses:

$$\sigma_{\xi\xi}(\xi,\eta,t) = \frac{\lambda}{f^2}\left(\frac{1-\eta^2}{\xi^2-1}\right)^{\frac{1}{2}}\Big[K\left(\frac{\xi^2-1}{1-\eta^2}\right)^{\frac{1}{2}}\frac{\partial^2\Phi}{\partial\xi^2} + \left(\frac{1-\eta^2}{\xi^2-1}\right)^{\frac{1}{2}}\frac{\partial^2\Phi}{\partial\eta^2}$$

$$+ (K-1)\frac{\partial^2\Psi}{\partial\xi\partial\eta} + \frac{K(1-\eta^2)+2\xi^2-1-\eta^2}{(\xi^2-\eta^2)(\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}\left(\xi\frac{\partial\Phi}{\partial\xi} - \eta\frac{\partial\Phi}{\partial\eta}\right) - \frac{(K-1)\xi}{\xi^2-\eta^2}\frac{\partial\Psi}{\partial\eta}$$

$$- \frac{(K-1)(\xi^2-1)\eta}{(1-\eta^2)(\xi^2-\eta^2)}\frac{\partial\Psi}{\partial\xi} + \frac{(K-1)\xi\eta\Psi}{(\xi^2-1)(1-\eta^2)}\Big] = P(t) \qquad (10)$$

where $P(t)$ is the function describing the normal pressure on the sphere $\xi + \eta = \frac{\epsilon}{f}$ as a function of time. For shear stresses:

$$\sigma_{\xi\eta}(\xi,\eta,t) = \frac{\mu(\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}{f^2(\xi^2-\eta^2)}\Big[2\frac{\partial^2\Phi}{\partial\xi\partial\eta} + \left(\frac{1-\eta^2}{\xi^2-1}\right)^{\frac{1}{2}}\frac{\partial^2\Psi}{\partial\eta^2}$$

$$- \left(\frac{\xi^2-1}{1-\eta^2}\right)\frac{\partial^2\Psi}{\partial\xi^2} + \frac{2\eta}{\xi^2-\eta^2}\frac{\partial\Phi}{\partial\xi} - \frac{2\xi}{\xi^2-\eta^2}\frac{\partial\Phi}{\partial\eta}$$

$$- \left(1 + \frac{\xi^2-1}{\xi^2-\eta^2} - \frac{1-\eta^2}{\xi^2-\eta^2}\right)\frac{\eta}{(\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}\frac{\partial\Psi}{\partial\eta}$$

$$- \left(1 + \frac{1-\eta^2}{\xi^2-\eta^2} - \frac{\xi^2-1}{\xi^2-\eta^2}\right)\frac{\xi}{(\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}\frac{\partial\Psi}{\partial\xi}$$

$$+ \left(\frac{2}{(\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}} + \frac{\xi^2}{(\xi^2-1)^{\frac{3}{2}}(1-\eta^2)^{\frac{1}{2}}} - \frac{\eta^2}{(\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{3}{2}}}\right)\Psi\Big] = 0 \qquad (11)$$

The wave equations (3), (7) and the boundary conditions (8), (9), (10), (11) completely specify the problem. Although the wave equations can be easily separated into ordinary differential equations whose solutions are spheroidal wave functions of zero order [2], it is extremely difficult to determine the solution satisfying all the boundary conditions. Therefore, the state of stress at a point in the spheroid at any instant cannot be determined by routine analytical methods.

A Method of Analysis

However, we shall investigate only the propagation of the incident dilatation waves and their reflection as well as their convergence. For convenience, assuming that the compressional waves as diverging from a sphere of radius ϵ to the entire space, the wave propagation is radially symmetric. Using spherical coordinates (1) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} = \frac{1}{C_1^2}\frac{\partial^2 u}{\partial t^2} \tag{12}$$

where u is the radial displacement.

The initial condition is

$$u = 0, \quad \frac{\partial u}{\partial t} = 0 \quad for \quad t = 0, \quad r \geq \epsilon \tag{13}$$

The boundary conditions are

$$\sigma_{rr}(r,t) = (\lambda + 2\mu)\frac{\partial u}{\partial r} + 2\lambda\frac{u}{r} = \begin{cases} P(t) & for\ t > 0,\ r = \epsilon \\ 0 & for\ t > 0,\ r \to \infty \end{cases} \tag{14}$$

For simplicity, assume that $P(t)$ is of the form

$$P(t) = A e^{-Bt} \tag{15}$$

where A and B are suitable constants. Then, using the Laplace transform and the initial condition (13), (12), and (14) can be put respectively as follows:

$$\frac{d^2\bar{u}}{dr^2}+\frac{2}{r}\frac{d\bar{u}}{dr}-\left(\frac{2}{r^2}+\frac{s^2}{c_1^2}\right)\bar{u}=0 \tag{16}$$

$$(\lambda+2\mu)\frac{d\bar{u}}{dr}+2\lambda\frac{\bar{u}}{r}=\begin{cases}\frac{A}{s+B} & \text{for } r=\epsilon\\ 0 & \text{for } r\to\infty\end{cases} \tag{17}$$

where

$$\bar{u}(r,s)\equiv\int_0^\infty u(r,t)e^{-st}dt \tag{18}$$

Solving (16) and substituting into (17) we obtain

$$\bar{u}(r,s)=-\frac{A\epsilon^3\left(1+\frac{rs}{c_1}\right)e^{-\frac{s}{c_1}(r-\epsilon)}}{4\mu\left[1+\frac{\epsilon s}{c_1}+\frac{\lambda+2\mu}{4\mu}\left(\frac{\epsilon s}{c_1}\right)^2\right](s+B)r^2} \tag{19}$$

The singularities are at

$$s=\begin{cases}-B\\ \dfrac{-2\mu\pm\left[(\lambda+2\mu)^2-4\mu^2\right]^{\frac{1}{2}}}{\lambda+2\mu}.\end{cases} \tag{20}$$

Inversion can be performed by calculating residues. For a specific case when Poisson's ratio is $\frac{1}{3}$, $\lambda=2\mu$ we can eventually obtain the inversion for $t\geq 0$

$$u(r,t)=\frac{A\epsilon^3}{4\mu r^2\left[1-\frac{B\epsilon}{c_1}+\left(\frac{B\epsilon}{c_1}\right)^2\right]}\left[\left(-1+\frac{Br}{c_1}\right)e^{-\frac{B\epsilon}{c_1}}e^{-B\left(t-\frac{r}{c_1}\right)}\right.$$

$$-\left(-1+\frac{Br}{c_1}\right)e^{-\frac{1}{2}\left(1+\frac{c_1t-r}{\epsilon}\right)}\cos\frac{3^{\frac{1}{2}}}{2}\left(1+\frac{c_1t-r}{\epsilon}\right)$$

$$\left.+3^{-\frac{1}{2}}\left[1-\frac{2r}{\epsilon}+\frac{B}{c_1}(r-2\epsilon)\right]e^{-\frac{1}{2}\left(1+\frac{c_1t-r}{\epsilon}\right)}\sin\frac{3^{\frac{1}{2}}}{2}\left(1+\frac{c_1t-r}{\epsilon}\right)\right] \tag{21}$$

Now let us consider the reflection of waves. The reflection of harmonic plane waves on the stress-free infinite plane is a problem which has a simple solution. It is a familiar result that when a dilatation wave is incident to a free plane surface both dilatation and shear waves are reflected. If the free surface is smooth and curved, we may divide the surface into infinitesimal area elements and regard each surface element as a plane surface. The law of reflection of plane waves on a plane boundary should then be valid for each surface element. The incident spherical waves may also be regarded as plane waves when the source of the wave is sufficiently far away (as compared with the dimension of the surface element) from the infinitesimal surface element. If the source is not far removed as is the case in this problem, the spherical wave can be expanded into plane waves using double Fourier integrals. Then the assumption made in this method is quite reasonable [3] .

Let A_1, A_2, A_3 denote respectively the displacement amplitude of the incident dilatation wave, that of the reflected dilatation wave, and that of the reflected shear wave. If α is the angle between the normal of the free surface and the direction of the incident wave or that of the reflected wave, and β is the angle between the normal of the free surface and the wave normal of the reflected shear wave, then the following equations must be satisfied.

$$\frac{\sin\alpha}{\sin\beta} = \left(\frac{\lambda+2\mu}{\mu}\right)^{\frac{1}{2}} = \frac{C_1}{C_2} \tag{22}$$

$$2(A_1 - A_2)\cos\alpha \sin\beta - A_3 \cos 2\beta = 0 \tag{23}$$

$$(A_1 + A_2)\cos 2\beta \sin\alpha - A_3 \sin\beta \sin 2\beta = 0 \tag{24}$$

These relationships are derived for plane harmonic waves of any frequency; hence, they may be applied to plane waves which are arbitrary functions of time. The displacement amplitude of an incident harmonic dilatation wave of frequency ω can be expressed as

$$\phi_1 = A_1 \sin\omega\left(t - \frac{x\cos\alpha + y\sin\alpha}{C_1}\right) \tag{25}$$

where x , y are local perpendicular coordinates with x-axis parallel to the outward normal at point of reflection on the spheroidal surface. The amplitude of the reflected dilatation waves can be expressed as

$$\phi_2 = A_2 \sin\omega\left(t - \frac{x\cos\alpha - y\sin\alpha}{C_1}\right) \tag{26}$$

For an arbitrary frequency ω the only difference between the functional forms of ϕ_1 and ϕ_2 is due to the difference between the directions of the wave normals. We conclude that when the dilatation wave $U_1(r, t)$ characterized by (21) is incident on a surface element with an incident angle α , the reflected dilatation wave in the neighborhood of the surface element is described by

$$U_2(\rho, t) = -\frac{A\epsilon^3 F(\alpha)}{4\mu(2a-\rho)^2\left[1-\frac{B\epsilon}{C_1}+\left(\frac{B\epsilon}{C_1}\right)^2\right]}\left[\left(-1+\frac{B(2a-\rho)}{C_1}\right)e^{\frac{B}{C_1}(2a-\epsilon)}e^{-B\left(t+\frac{\rho}{C_1}\right)}\right.$$

$$-\left(-1+\frac{B(2a-\rho)}{C_1}\right)e^{-\frac{1}{2}+\frac{a}{\epsilon}}\,e^{-\frac{1}{2}\left(\frac{C_1 t}{\epsilon}+\frac{\rho}{\epsilon}\right)}\cos\frac{3^{\frac{1}{2}}}{2}\left(\frac{C_1 t+\rho-2a}{\epsilon}+1\right)$$

$$\left.+3^{-\frac{1}{2}}\left(1-\frac{4a-2\rho}{\epsilon}+\frac{2B(a-\epsilon)}{C_1}-\frac{B\rho}{C_1}\right)e^{-\frac{1}{2}+\frac{a}{\epsilon}}\,e^{-\frac{C_1 t+\rho}{2\epsilon}}\sin\frac{3^{\frac{1}{2}}}{2}\left(\frac{C_1 t+\rho-2a}{\epsilon}+1\right)\right] \tag{27}$$

where ρ is the distance between the surface element and the second focus. The function $F(\alpha) = \frac{A_2}{A_1}$ can be calculated from (23) and (24) with $\lambda = 2\mu$. Thus, we obtain

$$F(\alpha) = \frac{\sin\alpha\sin 2\alpha\,(4-\sin^2\alpha)^{\frac{1}{2}} - 2(2-\sin^2\alpha)^2}{\sin\alpha\sin 2\alpha\,(4-\sin^2\alpha)^{\frac{1}{2}} + 2(2-\sin^2\alpha)^2}. \tag{28}$$

Taking spherical coordinates (ρ, θ, ϕ) with the origin at the second focus and the z-axis coinciding the axis of the spheroid, then for points on the spheroidal surface the following equation must be satisfied

$$\frac{2\rho}{\sin 2\alpha} - \frac{2a-\rho}{\sin\theta}$$

$$\text{or} \quad \alpha = \frac{1}{2}\sin^{-1}\left(\frac{2\rho}{2a-\rho}\sin\theta\right) \tag{29}$$

Substituting (29) in (27) we can obtain an expression for $U_2(\rho, \theta, t)$ Using this radial displacement function, the state of stress in the neighborhood of the second focus may be studied. That is, with $U_2(\rho, \theta, t)$ given over the free surface

$$\rho = \frac{a - f\cos\theta}{1 + \frac{f^2}{b^2}\sin^2\theta} \tag{30}$$

it is required to find the radial and tangential displacements $U_\rho(\rho, \theta, t)$ and $U_\theta(\rho, \theta, t)$ for small ρ, or for points (ρ, θ) in the neighborhood of the second focus.

It should be noted that the function $U_2(\rho, \theta, t)$ is only applicable to points on the spheroidal surface. Away from the spheroidal surface the interaction among waves from surface elements will change the wave shape. Formula (27), with the left-hand side replaced by $U_2(\rho, \theta, t)$, does not represent the propagation of a wave. However, the function $U_2(\rho, \theta, t)$ as a time-dependent boundary condition actually determines the state of stress near the second focus.

It is seen that the exact determination of the stress field is extremely difficult. However, some rough idea could be obtained among several relevant quantities. It is of some interest to know the strain energy, transmitted to the second focus, which contributes to the fracture around the second focus. If W_1 and W_2 are the strain energies corresponding to dilatational waves before and after reflection respectively, then it can be shown that

$$\frac{W_2}{W_1} = \left(\frac{A_2}{A_1}\right)^2 = [F(\alpha)]^2 \tag{31}$$

The rate of transmission of energy across any closed surface is obtained by integrating (1) as

$$\frac{\partial W}{\partial t} = \oiint \left\{ (\lambda+\mu)\nabla\cdot\underset{\sim}{u}\,\dot{\underset{\sim}{u}} + \mu\,\dot{\underset{\sim}{u}}\cdot\nabla\underset{\sim}{u} \right\}\cdot\underset{\sim}{n}\,ds \tag{32}$$

where $\underset{\sim}{n}$ is the outward drawn normal. [4] . Evaluating (32) when $\lambda = 2\mu$ the rate of energy transmission across the spherical surface $r=\epsilon$

$$\frac{\partial W_1}{\partial t} = 8\pi\mu\epsilon^2 \frac{\partial u(\epsilon,t)}{\partial t}\left[2\frac{\partial u}{\partial r}(\epsilon,t) + 3\frac{u(\epsilon,t)}{\epsilon}\right] \tag{33}$$

Similarly, the rate of energy transmission to the second focus at the instant $t+\frac{1}{C_1}(2a-\epsilon)$ is obtained by evaluating (32) over the spheroidal surface as

$$\frac{\partial W_2(t+\frac{2a-\epsilon}{C_1})}{\partial t} = 4\pi\mu\epsilon^2\int_0^{\pi}\frac{\partial u}{\partial t}\left(2\frac{\partial u}{\partial r}+\frac{3u}{\epsilon}\right)$$

$$\left[F\left(\frac{1}{2}\sin^{-1}\frac{2f\sin\psi}{2a-\dfrac{a-f\cos\psi}{1+\frac{f^2}{b^2}\sin^2\psi}}\right)\right]^2 \sin\psi\,d\psi \tag{34}$$

Integration of (34) over a period of time, say between t_1 and t_2, gives the total $W_2(t_1,t_2)$ transmitted during this period. Since all paths of the dilatational waves between two foci are equal, the second focus will be under uniform hydrostatic tension. We can then assume that a small neighborhood near the second focus is also under hydrostatic tension. Let δ be the radius of the fracture and E the amount of strain energy per unit volume of material under fracture, then

$$W_2 \cong \frac{4}{3} \pi \delta^3 E$$

or

$$\delta \cong \left(\frac{3}{4} \frac{W_2}{\pi E}\right)^{\frac{1}{3}} \tag{35}$$

In case E is not known, and W_m is the maximum value of $W_2(t, t + \frac{2\delta}{c_1})$, $t > 0$ and W_0 is the minimum value below which no fracture will occur, then, for the case $\lambda = 2\mu$

$$W_m - W_0 = \iiint_{r \le \delta} \sigma_{ij}\, e_{ij}\, dv \cong \frac{\pi}{2\mu} p^2 \delta^3$$

where p is hydrostatic pressure under which fracture occurs. Or

$$\delta \cong \left[\frac{2\mu}{\pi} \frac{(W_m - W_0)}{p^2}\right]^{\frac{1}{3}}; \qquad W_m \ge W_0 \tag{36}$$

Relation (36) gives a theoretical estimate on the fracture for a given p and W_m. Fig 1 gives the relationship between $W_m - W_0$ and δ. It is interesting to note that for constant W_m the larger the value of p the smaller is δ. This relation can also be used in the estimation of hydrostatic pressure p for fracture if δ is measured. The theoretical value of W_m is overestimated here since a portion of the energy must be spent in inducing fracture near the first focus. If (36) is used in evaluating p it can be shown that p will be underestimated. Another matter to be noticed is that before (36) can be evaluated, the magnitude A in (15) must be determined. This by no means is easy. However, it is reasonable to assume that A should be the same for the same amount of explosive used. This would provide a way to study and compare with experimental results. Some experimental data are shown in the following section.

(b) Fracture under a State of Radial Tension

The strength and fracture of a solid under a state of tri-axial tension has long been of interest to many research workers [5] . This is particularly true in finding practical methods for obtaining experimental data. One method of using reflected intense mechanical waves and focusing them to a point in a prolate spheroid is found to be quite successful under suitable conditions. By exploding lead azide at one focus of a prolate spheroid made of polystyrene or polymethyl methacrylate materials extremely high compressional waves can be created. These waves radiate out from the vicinity of the focal point and converge to the second focus after being reflected from the free spheroidal surface. As the reflected waves are tensile in nature, fracture occurs when they converge to one point for a sufficient length of time. Figs 2 and 3 illustrate two fractured specimens. Fig 4 shows some experimental data. This compares fairly well with theoretical results if proportional reduction of the strain energy from explosive charges is assumed.

By comparison of the theoretical and experimental results, it appears that

$$p = \left(\frac{2\mu}{\pi} \frac{W_m - W_o}{\delta^3}\right)^{\frac{1}{2}}$$

which is constant as $(W_m - W_o)/\delta^3$ is found to be invariant for almost any point on the experimental curve. If the absolute magnitude can be determined, then the internal stress p required for fracture under a state of hydrostatic tension will be obtained.

It might also be of importance to mention that the inception of a crack at the second focus of the prolate spheroid is essentially resulted by a hydrostatic tension. However, the state of stress will not be hydrostatic when the volume of fracture sphere becomes appreciable. This is so not only because of the geometry of the prolate spheroid but also the incomplete reflection of the pulses from the spheroidal surface.

(c) Three-Dimensional Crack Propagation in a Transversely Isotropic Viscoelastic Medium.

Introduction

After the initiation of a crack in a solid, the studying of the propagation of the crack in fracture processes becomes necessary. The basic knowledge learned will be important in the progress of fracture mechanics. In general, material bodies are predominantly viscoelastic, the molecules of a solid become oriented especially in the neighborhood of a crack and the medium is likely to be transversely isotropic as a result of homogeneous deformation. It seems highly desirable, in studying fracture mechanics, to consider the crack propagation problem by taking into consideration the following important points: 1. the viscoelastic behavior of material bodies; 2. the anisotropic nature of deformed solids; and 3. the three-dimensional features of the state of stress and displacement fields as well as other related quantities in the problem of crack propagation.

In the following sections the effects of a finite penny shaped crack on the stress and displacement fields in an infinite transversely isotropic viscoelastic medium are investigated. Relations between the pressure distribution and the opening of the crack are derived. Also, the most likely shape of the opened-up crack is analyzed. Knowing the shape of the crack, we obtain results for an expanding crack in a viscoelastic medium. Numerical results are also given for a physically realistic material for which relaxation data are based on certain known results for oriented materials.

Preliminary Background Considerations

The effect of a crack on the state of stress and displacement can be investigated as the solution of a mixed boundary value problem for a semi-infinite medium. As a preliminary, let us assume that the medium is transversely isotropic, viscoelastic and has properties symmetrical about the χ_3 axis in an arbitrary coordinate system. Consider the case that the crack propagation is not significantly affected by inertia forces of the medium, and in the absence of body forces, the equation of motion becomes

$$\frac{\partial \sigma_{ij}(\chi,t)}{\partial \chi_j} = 0 \tag{37}$$

where $\sigma_{ij}(\chi,t)$ is a symmetric stress tensor. All indicial notations range over the integers 1,2,3 and summation over repeated indices is implied. Also χ stands for the triplet of coordinates χ_1, χ_2, χ_3.

The constitutive equations for an anisotropic viscoelastic medium can in general be expressed in the following form for large nonlinear deformations.

$$\begin{aligned}\sigma_{ij}(\chi,t) = {} & \int_0^t C_{ijkl}(\chi, t-\tau)\,\frac{\partial \varepsilon_{kl}(\chi,\tau)}{\partial \tau}\,d\tau \\ & + \int_0^t\int_0^t D_{ijkl}(\chi, t-\tau_1, t-\tau_2)\,\frac{\partial \varepsilon_{kl}(\chi,\tau_1)}{\partial \tau_1}\,\frac{\partial \varepsilon_{kl}(\chi,\tau_2)}{\partial \tau_2}\,d\tau_1 d\tau_2 \\ & + \int_0^t\int_0^t\int_0^t E_{ijkl}(\chi, t-\tau_1, t-\tau_2, t-\tau_3)\,\frac{\partial \varepsilon_{kl}(\chi,\tau_1)}{\partial \tau_1}\,\frac{\partial \varepsilon_{kl}(\chi,\tau_2)}{\partial \tau_2}\,\frac{\partial \varepsilon_{kl}(\chi,\tau_3)}{\partial \tau_3}\,d\tau_1 d\tau_2 d\tau_3\end{aligned} \tag{38}$$

where $C_{ijkl}(x,t)$, $D_{ijkl}(x,t)$ and $E_{ijkl}(x,t)$ are anisotropic functions of relaxation moduli. It is assumed that the body is in its undeformed state and $C_{ijkl}(x,t)=0$ when $t \leq 0$, and $C_{ijkl}(x,t)>0$ when $t>0$. Under suitably restricted conditions it may be possible to determine the general response by using the same relaxation functions $C_{ijkl}(x,t)$. Or, $D_{ijkl}(x,t)$ and $E_{ijkl}(x,t)$ may be replaced by functions of $C_{ijkl}(x,t)$. It is naturally expected that for large deformations the contributions from the double and triple integrals will not be negligibly small. However, for relatively small deformations, the first integral alone will be sufficient for obtaining the required information [3]. To illustrate the method of analysis we shall consider the linear case so that the Laplace transform technique may be employed. In this case small strains $\varepsilon_{ij}(x,t)$ and displacements $u_i(x,t)$ will be related as follows

$$2\,\varepsilon_{ij}(x,t)=\frac{\partial u_i(x,t)}{\partial x_j}+\frac{\partial u_j(x,t)}{\partial x_i} \tag{39}$$

Use of the Laplace transform $\bar{\Phi}(s)$ of a function defined [7] by

$$\bar{\Phi}(s)=\int_0^\infty \phi(t)e^{-st}dt \tag{40}$$

will reduce the first order integral relations (38) to linear algebraic relations. Limiting ourselves to homogeneous transversely isotropic media symmetrical about x_3-axis, (38) through the use of (40) reduces to

$$\bar{\sigma}_{11}(x,s) = s\bar{C}_{11}(s)\bar{\varepsilon}_{11}(x,s) + s\bar{C}_{12}(s)\bar{\varepsilon}_{22}(x,s) + s\bar{C}_{13}(s)\bar{\varepsilon}_{33}(x,s)$$

$$\bar{\sigma}_{22}(x,s) = s\bar{C}_{12}(s)\bar{\varepsilon}_{11}(x,s) + s\bar{C}_{11}(s)\bar{\varepsilon}_{22}(x,s) + s\bar{C}_{13}(s)\bar{\varepsilon}_{33}(x,s)$$

$$\bar{\sigma}_{33}(x,s) = s\bar{C}_{13}(s)\bar{\varepsilon}_{11}(x,s) + s\bar{C}_{13}(s)\bar{\varepsilon}_{22}(x,s) + s\bar{C}_{33}(s)\bar{\varepsilon}_{33}(x,s)$$

$$\bar{\sigma}_{23}(x,s) = 2s\bar{C}_{44}(s)\bar{\varepsilon}_{23}(x,s)$$

$$\bar{\sigma}_{31}(x,s) = 2s\bar{C}_{44}(s)\bar{\varepsilon}_{31}(x,s)$$

$$\bar{\sigma}_{12}(x,s) = s[\bar{C}_{11}(s) - \bar{C}_{12}(s)]\bar{\varepsilon}_{12}(x,s) \tag{41}$$

where $\bar{C}_{mn}$ $(mn = 1 \text{ to } 6)$ are used, replacing $\bar{C}_{ijkl}$ for convenience and simplicity.

Using cylindrical coordinates (r, θ, z) with z along the x-axis, it has been shown that (results to be published) all the equations (37), (38), (39), (41) are satisfied for a symmetrical problem if

$$\bar{u}_r(r,z,s) = \sum_{i=1}^{2} \bar{\chi}_i(r,z,s) + z\sum_{i=1}^{2} \bar{\alpha}_i(s)\frac{\partial \bar{\Psi}_i(r,z,s)}{\partial r}$$

$$\bar{u}_\theta(r,z,s) = 0 \tag{42}$$

$$\bar{u}_z(r,z,s) = \sum_{i=1}^{2} \bar{\beta}_i(s)\bar{\Psi}_i(r,z,s) + z\sum_{i=1}^{2} \bar{\mu}_i(s)\frac{\partial \bar{\Psi}_i(r,z,s)}{\partial z}$$

where $\bar{\Psi}_i(r,z,s)$ satisfies

$$\frac{\partial^2 \bar{\Psi}_i(r,z,s)}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Psi}_i(r,z,s)}{\partial r} + \bar{\eta}_i(s)\frac{\partial^2 \bar{\Psi}_i(r,z,s)}{\partial z^2} = 0 \tag{43}$$

and $$\frac{\partial \bar{\chi}_i(r,z,s)}{\partial z} = \frac{\partial \bar{\Psi}_i(r,z,s)}{\partial r}$$

$$\frac{\partial \overline{\chi}_i(r,z,s)}{\partial r} + \frac{\overline{\chi}_i(r,z,s)}{r} + \overline{\eta}_i(s)\frac{\partial \overline{\Psi}_i(r,z,s)}{\partial z} = 0 \qquad (i=1,2) \tag{44}$$

In (42) and (43) $\overline{\eta}_i(s)$, $\overline{\alpha}_i(s)$, $\overline{\beta}_i(s)$, $\overline{\mu}_i(s)$ and $\overline{m}_i(s)$ are defined as in the following equations

$$\overline{C}_{11}(s)\overline{C}_{44}(s)\overline{\eta}^2(s) + [(\overline{C}_{13}(s)+\overline{C}_{44}(s))^2 - \overline{C}_{44}^2(s) - \overline{C}_{11}(s)\overline{C}_{33}(s)]\,\overline{\eta}(s) + \overline{C}_{44}(s)\overline{C}_{33}(s) = 0$$

$$\overline{\alpha}_i(s) = \frac{[\overline{C}_{33}(s)-\overline{C}_{44}(s)\overline{\eta}_i(s)][\overline{C}_{11}(s)\overline{\eta}_i(s)-\overline{C}_{44}(s)] - \overline{\eta}_i(s)\overline{C}^2(s)}{\{[\overline{C}(s)\overline{m}_i(s)+2\overline{C}_{44}(s)][\overline{C}_{33}(s)-\overline{C}_{44}(s)\overline{\eta}_i(s)] - \overline{C}(s)[2\overline{C}_{33}(s)\overline{m}_i(s)-\overline{C}(s)\overline{\eta}_i(s)]\}}$$

$$\overline{\beta}_i(s) = \frac{\{[\overline{C}_{11}(s)\overline{\eta}_i(s)-\overline{C}_{44}(s)][2\overline{C}_{33}(s)\overline{m}_i(s)-\overline{C}(s)\overline{\eta}_i(s)] - \overline{C}(s)\overline{\eta}_i(s)[\overline{C}(s)\overline{m}_i(s)+2\overline{C}_{44}(s)]\}}{\{\overline{C}(s)[2\overline{C}_{33}(s)\overline{m}_i(s)-\overline{C}(s)\overline{\eta}_i(s)] - [\overline{C}(s)\overline{m}_i(s)+2\overline{C}_{44}(s)]\cdot[\overline{C}_{33}(s)-\overline{C}_{44}(s)\overline{\eta}_i(s)]\}}$$

$$\overline{\mu}_i(s) = \overline{m}_i(s)\,\overline{\alpha}_i(s) \qquad \text{(no summation)}$$

where $\overline{C}(s) = \overline{C}_{13}(s) + \overline{C}_{44}(s)$

$$\overline{m}_i(s) = \frac{\overline{C}_{11}(s)\overline{\eta}_i(s) - \overline{C}_{44}(s)}{\overline{C}(s)} \tag{45}$$

Substituting (42) into (41) we get

$$\bar{\sigma}_{rr}(r,z,s)=\sum_{K=1}^{2}\{s\bar{C}_{13}(s)[\bar{\beta}_K(s)+\bar{\mu}_K(s)]-s\bar{C}_{11}(s)\bar{\eta}_K(s)\}\frac{\partial\bar{\Psi}_K(r,z,s)}{\partial z}$$

$$-s\sum_{K=1}^{2}[\bar{C}_{11}(s)-\bar{C}_{12}(s)]\bar{\eta}_K(s)\frac{\bar{\chi}_K(r,z,s)}{r}+z\sum_{K=1}^{2}\{s\bar{C}_{13}(s)\bar{\mu}_K(s)\frac{\partial^2\bar{\Psi}_K(r,z,s)}{\partial z^2}$$

$$-s\bar{C}_{11}(s)\bar{\alpha}_K(s)\bar{\eta}_K(s)\frac{\partial^2\bar{\Psi}_K(r,z,s)}{\partial r\partial z}-\frac{s[\bar{C}_{11}(s)-\bar{C}_{12}(s)]}{r}\bar{\alpha}_K(s)\frac{\partial\bar{\chi}_K(r,z,s)}{\partial z}\}$$

$$\bar{\sigma}_{\theta\theta}(r,z,s)=\sum_{K=1}^{2}\{[s\bar{C}_{13}(s)[\bar{\beta}_K(s)+\bar{\mu}_K(s)]-s\bar{C}_{12}(s)\bar{\eta}_K(s)]\frac{\partial\bar{\Psi}_K(r,z,s)}{\partial z}$$

$$+\frac{s[\bar{C}_{11}(s)-\bar{C}_{12}(s)]}{r}\bar{\chi}_K(r,z,s)\}+z\sum_{K=1}^{2}\{s\bar{C}_{13}(s)\bar{\mu}_K(s)\frac{\partial^2\bar{\Psi}_K(r,z,s)}{\partial z^2}$$

$$-s\bar{C}_{12}(s)\bar{\eta}_K(s)\bar{\alpha}_K(s)\frac{\partial^2\bar{\Psi}_K(r,z,s)}{\partial r\partial z}+\frac{s[\bar{C}_{11}(s)-\bar{C}_{12}(s)]}{r}\bar{\alpha}_K(s)\frac{\partial\bar{\chi}_K(r,z,s)}{\partial z}$$

$$\bar{\sigma}_{zz}(r,z,s)=\sum_{K=1}^{2}[s\bar{C}_{33}(s)[\bar{\beta}_K(s)+\bar{\mu}_K(s)]-s\bar{C}_{13}(s)\bar{\eta}_K(s)]\frac{\partial\bar{\Psi}_K(r,z,s)}{\partial z}$$

$$+z\sum_{K=1}^{2}[s\bar{C}_{33}(s)\bar{\mu}_K(s)-s\bar{C}_{13}(s)\bar{\eta}_K(s)\bar{\alpha}_K(s)]\frac{\partial^2\bar{\Psi}_K(r,z,s)}{\partial z^2}$$

$$\bar{\sigma}_{rz}(r,z,s)=\sum_{K=1}^{2}s\bar{C}_{44}(s)[1+\bar{\alpha}_K(s)+\bar{\beta}_K(s)]\frac{\partial\bar{\Psi}_K(r,z,s)}{\partial r}$$

$$+z\sum_{K=1}^{2}s\bar{C}_{44}(s)[\bar{\alpha}_K(s)+\bar{\mu}_K(s)]\frac{\partial^2\bar{\Psi}_K(r,z,s)}{\partial r\partial z}$$

$$\bar{\sigma}_{r\theta}(r,z,s)=\bar{\sigma}_{\theta z}(r,z,s)=0 \tag{46}$$

With these definitions, any radially symmetrical boundary value problem can be solved once proper functions $\overline{\psi}_i(r,z,s)$ are found. It can be shown that (43) is satisfied by

$$\overline{\psi}_i(r,z,s) = [r^2 + \overline{\omega}_k^2(s) z^2]^{-\frac{1}{2}} \quad (k=1,2) \quad \text{where } \overline{\eta}_i(s) = \overline{\omega}_i(s).$$

Then successive integration with respect to z produces a series of harmonic functions, usually known as first, second........... logarithmic potentials [8]. If $\overline{\omega}_k(s)z$ is replaced by $[\overline{\omega}_k(s)z + i\xi]$ where ξ is a real variable and $i=(-1)^{\frac{1}{2}}$ the new function $\underset{\sim}{\overline{\psi}_k}(r,z,s) = \{r^2 + [\overline{\omega}_k(s)z + i\xi]^2\}^{-\frac{1}{2}}$ so created is also harmonic because derivatives with respect to $\overline{\omega}_k(s)z$ are the same as those with respect to $[\overline{\omega}_k(s)z + i\xi]$. The function $\underset{\sim}{\overline{\psi}_k}(r,z,s)$ is complex and its real and imaginary parts are also harmonic. A judicious choice of one of these logarithmic potentials will enable us to solve the problems in which the proper boundary conditions are stipulated [9].

Formulation of the Problem

With this preliminary information, the problem of a penny shaped crack in an infinite medium can be reduced to that of a half-space with the boundary conditions

on $z=0$ plane

$$\begin{aligned} \sigma_{zz}(r,0,t) &= -p(r,t) && 0 \le r \le R \\ u_z(r,0,t) &= 0 && R \le r < \infty \\ \sigma_{rz}(r,0,t) &= 0 && 0 \le r < \infty \end{aligned} \tag{47}$$

where R is the radius of the crack. If the crack is expanding, then $R=R(t)$. However, we will first solve the problem of an unexpanding crack and then extend the results to the case of an expanding crack.

With a little examination it follows that the proper functions $\bar{\psi}_K(r,z,s)$ in this case, i.e., R = constant, are

$$\bar{\psi}_K(r,z,s) \sim \ln\left\{\bar{\omega}_K(s)z \pm i\xi + [(\bar{\omega}_K(s)z \pm i\xi)^2 + r^2]^{\frac{1}{2}}\right\}$$

Generalizing to take into account the variation on the crack, we choose

$$\bar{\psi}_K(r,z,s) = Re\int_0^R \bar{h}_K(\xi,s)\left\{\ln[\bar{\omega}_K(s)z + i\xi + ([\bar{\omega}_K(s)z + i\xi]^2 + r^2)^{\frac{1}{2}}] + \ln[\bar{\omega}_K(s)z - i\xi + ([\bar{\omega}_K(s)z - i\xi]^2 + r^2)^{\frac{1}{2}}]\right\}d\xi \tag{48}$$

then

$$\bar{\chi}_K(r,z,s) = Re\int_0^R \frac{\bar{h}_K(\xi,s)}{\bar{\omega}_K(s)r}\left\{2\bar{\omega}_K(s)z - ([\bar{\omega}_K(s)z + i\xi]^2 + r^2)^{\frac{1}{2}} - ([\bar{\omega}_K(s)z - i\xi)^2 + r^2)^{\frac{1}{2}}\right\}d\xi \tag{49}$$

where $\bar{h}_K(\xi,s)$ are arbitrary functions to be determined from the boundary conditions. Substituting (48) into (47) through the use of (46) we get

$$\int_0^r \frac{\bar{h}_1(\xi,s)\,d\xi}{(r^2-\xi)^{\frac{1}{2}}} = -\frac{\bar{p}(r,s)}{2\bar{A}(s)} \tag{50}$$

$$\int_0^R \bar{h}_1(\xi,s)\,d\xi = 0 \tag{51}$$

and

$$\bar{h}_2(\xi,s) = -\frac{1+\bar{\alpha}_1(s)+\bar{\beta}_1(s)}{1+\bar{\alpha}_2(s)+\bar{\beta}_2(s)}\,\bar{h}_1(\xi,s) \tag{52}$$

where

$$\bar{A}(s) = s\bar{\omega}_1(s)\left\{\bar{C}_{33}(s)[\bar{\beta}_1(s)+\bar{\mu}_1(s)] - \bar{C}_{13}(s)\bar{\eta}_1(s)\right\}\cdot \cdot\left\{1 - \frac{1+\bar{\alpha}_1(s)+\bar{\beta}_1(s)}{1+\bar{\alpha}_2(s)+\bar{\beta}_2(s)}\cdot\frac{\bar{\omega}_2(s)}{\bar{\omega}_1(s)}\cdot\frac{\bar{C}_{33}(s)[\bar{\beta}_2(s)+\bar{\mu}_2(s)] - \bar{C}_{13}(s)\bar{\eta}_2(s)}{\bar{C}_{33}(s)[\bar{\beta}_1(s)+\bar{\mu}_1(s)] - \bar{C}_{13}(s)\bar{\eta}_1(s)}\right\} \tag{53}$$

It is seen that (50) is a singular integral equation of the Abel type. Multiply both sides by $r/(\rho^2-r^2)^{\frac{1}{2}}$ and integrate with respect to r over $(0,\rho)$, and eventually we can obtain

$$\frac{\pi}{2}\int_0^\rho \bar{h}_1(\xi,s)\,d\xi = -\frac{1}{2\bar{A}(s)}\int_0^\rho \frac{r\bar{p}(r,s)\,dr}{(\rho^2-r^2)^{\frac{1}{2}}}$$

Differentiating with respect to ρ

$$\bar{h}_1(\rho,s) = -\frac{1}{\pi\bar{A}(s)}\frac{d}{d\rho}\int_0^\rho \frac{r\bar{p}(r,s)\,dr}{(\rho^2-r^2)^{\frac{1}{2}}}$$

Now $\bar{h}_1(\rho,s)$ has to satisfy (51) in order that the displacement boundary condition is satisfied. It can be shown that all the boundary conditions are satisfied if we modify the above equation by adding a Dirac Delta function as

$$\bar{h}_1(\rho,s) = -\frac{1}{\pi\bar{A}(s)}\left\{\frac{d}{d\rho}\int_0^\rho \frac{r\bar{p}(r,s)\,dr}{(\rho^2-r^2)^{\frac{1}{2}}} - C\,\delta(R-\rho)\right\} \tag{54}$$

where

$$C = \left[\int_0^\rho \frac{r\bar{p}(r,s)\,dr}{(\rho^2-r^2)^{\frac{1}{2}}}\right]_{\rho=0}^{\rho=R}$$

and $\int_0^R \delta(R-r)\,dr = 1$

Substituting (54) into (52), (48) and (49) $\bar{h}_2(\xi,s)$, $\bar{\psi}_K(r,z,s)$ and $\bar{\chi}_K(r,z,s)$ may be computed. Both $\bar{\psi}_K(r,z,s)$, $\bar{\chi}_K(r,z,s)$ go to zero as $(r^2+z^2)^{\frac{1}{2}} \to \infty$.

On the $z=0$ plane

$$\bar{\Psi}_K(r,0,s) = 2\int_r^R \bar{h}_K(\xi,s)\, \ln\frac{[\xi+(\xi^2+r^2)^{\frac{1}{2}}]d\xi}{r} \qquad 0 \le r \le R$$

$$= 0 \qquad R \le r < \infty$$

$$\frac{\partial\bar{\Psi}_K(r,0,s)}{\partial z} = 2\int_0^r \frac{\bar{\omega}_K(s)\bar{h}_K(\xi,s)d\xi}{(r^2-\xi^2)^{\frac{1}{2}}} \qquad 0 \le r \le R$$

$$= 2\int_0^R \frac{\bar{\omega}_K(s)\bar{h}_K(\xi,s)d\xi}{(r^2-\xi^2)^{\frac{1}{2}}} \qquad R \le r < \infty$$

$$\bar{\chi}_K(r,0,s) = -2\int_0^r \frac{\bar{h}_K(\xi,s)}{\bar{\omega}_K(s)r}\,(r^2-\xi^2)^{\frac{1}{2}}d\xi \qquad 0 \le r \le R$$

$$= -2\int_0^R \frac{\bar{h}_K(\xi,s)}{\bar{\omega}_K(s)r}\,(r^2-\xi^2)^{\frac{1}{2}}d\xi \qquad R \le r < \infty$$

(55)

Then, substituting (55) into (46) we get, on $z=0$ plane

$$\bar{u}_z(r,0,s) = 2\,\bar{B}(s)\int_r^R \bar{h}_1(\xi,s)\, \ln\frac{\xi+(\xi^2-r^2)^{\frac{1}{2}}}{r}d\xi \qquad 0 \le r \le R$$

$$= 0 \qquad R \le r < \infty$$

$$\bar{\sigma}_{zz}(r,0,s) = -\,\bar{p}(r,s) \qquad 0 \le r \le R$$

$$= 2\bar{A}(s)\int_0^R \frac{\bar{h}_1(\xi,s)d\xi}{(r^2-\xi^2)^{\frac{1}{2}}} \qquad R \le r < \infty$$

$$\bar{U}_r(r,0,s) = -2\sum_{k=1}^{2}\int_0^r \frac{\bar{h}_k(\xi,s)}{\bar{\omega}_k(s)\,r}(r^2-\xi^2)^{\frac{1}{2}}\,d\xi \qquad 0\le r\le R$$

$$= -2\sum_{k=1}^{2}\int_0^R \frac{\bar{h}_k(\xi,s)}{\bar{\omega}_k(s)\,r}(r^2-\xi^2)^{\frac{1}{2}}\,d\xi \qquad R\le r<\infty \tag{56}$$

where

$$\bar{B}(s) = \bar{\beta}_1(s) - \frac{1+\bar{\alpha}_1(s)+\bar{\beta}_1(s)}{1+\bar{\alpha}_2(s)+\bar{\beta}_2(s)}\,\bar{\beta}_2(s) \tag{57}$$

Similarly, the rest of the quantities can be computed. From the results obtained so far it is apparent that the displacement $\bar{U}_z(r,0,s)$ on the crack surface is related to the applied loading through $\bar{h}_k(\xi,s)$. In other words, the displacement of the crack surface must be consistent with the applied loading. Then the applied load function $\bar{p}(r,s)$ required for producing a desirable displacement $\bar{U}_z(r,0,s)$ can be easily computed. To obtain such a relation, differentiate the first of (56) with respect to r, then

$$\frac{r}{\bar{B}(s)}\,\frac{d\bar{U}_z(r,0,s)}{dr} = -\int_r^R \frac{\xi\,\bar{h}_1(\xi,s)\,d\xi}{(\xi^2-r^2)^{\frac{1}{2}}}$$

This is also a singular integral equation which may be solved for $\bar{h}_1(\xi,s)$, finally as

$$\bar{h}_1(r,s) = \frac{2}{\pi\bar{B}(s)}\left[\frac{1}{r}\frac{d}{dr}\int_r^R \frac{d\bar{U}_z(\rho,0,s)}{d\rho}\,\frac{\rho^2\,d\rho}{(\rho^2-r^2)^{\frac{1}{2}}} - \bar{C}\,\delta(R-r)\right] \tag{58}$$

where

$$\bar{C} = \int^{R} \frac{dr}{r} \frac{d}{dr} \int_{r}^{R} \frac{d\bar{U}_z(\rho,0,s)}{d\rho} \frac{\rho^2 d\rho}{(\rho^2 - r^2)^{\frac{1}{2}}}$$

Substitute (58) into (50), then

$$\bar{p}(r,s) = -\frac{4\bar{A}(s)}{\pi \bar{B}(s)} \int_0^r \frac{d\xi}{\xi(r^2-\xi^2)^{\frac{1}{2}}} \frac{d}{d\xi} \int_{\xi}^{R} \frac{d\bar{U}_z(\rho,0,s)}{d\rho} \frac{\rho^2 d\rho}{(\rho^2-\xi^2)^{\frac{1}{2}}} \tag{59}$$

This equation is equivalent to the first of (56). A similar relation between the crack opening $\bar{U}_z(r,0,s)$ and the total load $\bar{P}(s)$ can be obtained as

$$\bar{P}(s) = -\frac{8\bar{A}(s)}{\bar{B}(s)} \int_0^R r\,dr \int_0^r \frac{d\xi}{\xi(r^2-\xi^2)^{\frac{1}{2}}} \frac{d}{d\xi} \int_{\xi}^{R} \frac{d\bar{U}_z(\rho,0,s)}{d\rho} \frac{\rho^2 d\rho}{(\rho^2-\xi^2)^{\frac{1}{2}}}$$

$$= 2\pi \int_0^R r\bar{p}(r,s)\,dr \tag{60}$$

The relation expressed in (59) is important because it gives a surface loading distribution for any prescribed crack opening. Sneddon [10] has computed similar results for a parabolic opening of the crack in an isotropic elastic medium. Use of relations (59) and (60) together with certain energy considerations will enable us to predict the most likely shape of the opened-up crack, at least for the elastic transversely isotropic medium. To obtain results in the real time domain inversion of the above equations must be performed. A mere examination of the expressions $\bar{\alpha}(s)$, $\bar{\beta}(s)$, etc. and $\bar{B}(s)$ and $\bar{A}(s)$ will indicate that an exact inversion of these expressions in many cases may be extremely difficult. However, the limiting solutions at

$t=0$ and $t=\infty$ can be easily obtained. To extend the solution for an entire time scale, approximate inversion methods can be employed and the solution expressed as the sum of exponential functions. This is expounded in a following section regarding a moving crack in a viscoelastic medium.

Most Likely Shape of the Opened-Up Crack

In this section alone assume that the medium is transversely isotropic as before but elastic. Then the quantities without overhead bars denote the corresponding quantities for the elastic medium. Let us assume that

$$U_z(r,0) = b_0 + b_1 r + b_2 r^2 + \cdots + b_N r^N \tag{61}$$

where b_n are constants to be determined. N is the number to be chosen depending upon the accuracy required. The above equation is quite general since any smooth function can be expressed in this form. Continuity of $U_z(r,0)$ requires that

$$U_z(R,0) \equiv b_0 + b_1 R + b_2 R^2 + \cdots + b_N R^N = 0 \tag{62}$$

which expresses b_0 in terms of other b_n and R. For the elastic medium R is determined from the Griffith's criterion of fracture [11]

$$\delta(W-U) = 0$$

where W is free energy of an elastic solid and U the surface energy. Then

$$W = \int_0^R 2\pi r\, p(r)\, U_z(r,0)\, dr$$

$$= -\frac{8A}{\delta} \sum_{n=0}^{N} \sum_{m=1}^{N} n\, b_n b_m\, G_{mn}(R)$$

where

$$G_{mn}(R)=\int_0^R r^{m+1}dr\int_0^r \frac{d\xi}{\xi(r^2-\xi^2)^{\frac{1}{2}}}\frac{d}{d\xi}\int_\xi^R \frac{\rho^{n+1}d\rho}{(\rho^2-\xi^2)^{\frac{1}{2}}} \tag{63}$$

and

$$U=2\pi R^2 T \tag{64}$$

where T is the surface tension of the material. It can then be stipulated that the critical values of R and b_n are those which minimize the energy balance. The minimum is given by

$$\frac{\partial(W-U)}{\partial R}=0$$

$$\frac{\partial(W-U)}{\partial b_n}=0 \quad ; \quad n=1,2,\cdots N \tag{65}$$

which give rise to equations

$$\frac{8A}{B}\sum_{n=0}^{N}\sum_{m=1}^{N} n b_n b_m \frac{dG_{mn}(R)}{dR}+4\pi R T=0$$

and

$$\sum_{m=1}^{N} b_m G_{mn}(R)=0 \quad ; \quad n=1,2,\cdots N \tag{66}$$

These are $(N+1)$ equations for R and N number of b_n unknowns and can be solved. Substituting them in (59) and (61) we obtain the most likely shape of the opened-up crack and the corresponding consistent pressure distribution. By increasing the number (61) will approach the exact displacement, however, the computations become increasingly difficult. It may be reasonable to expect that the most likely shape of the opened-up crack in a viscoelastic medium will have a somewhat similar form.

Moving Crack in Viscoelastic Medium

Now we will consider the same problem defined by (47) when $R = R(t)$ where $R(t)$ is a function of time t. Also we can formulate the same problem by prescribing a displacement, consistent with the loading $p(r,t)$, on the crack surface. The advantage in this case is that we can satisfy the boundary conditions more easily. Then on $z = 0$ plane

$$\begin{aligned} \sigma_{rz}(r,0,t) &= 0 & 0 \le r < \infty \\ u_z(r,0,t) &= W(r,t) & 0 \le r \le R(t) \\ &= 0 & R(t) \le r < \infty \end{aligned} \qquad (67)$$

We have shown in the early formulation of the problem that if R = constant the crack problem as formulated above can be solved easily. If we define a function $W_m(r,t)$ on $0 \le r \le R_m$ where R_m is a constant such that

$$\begin{aligned} W_m(r,t) &= W(r,t) & 0 \le r \le R(t) \\ &= 0 & R(t) \le r \le R_m. \end{aligned} \qquad (68)$$

then we can use the technique which has been used by Lee and Radok [12] in contact problems and generalized by Graham [13], to solve this problem with bounding conditions given in (67) in the case only when $R(t)$ is an increasing function. If we restrict the validity of the solution to $0 \leq t \leq T$ such that $R(t) \leq R_m \leq R(T)$, then the usual difficulties of using the Laplace transform for moving boundary problems will be overcome. Since T is an arbitrary number, the validity of the solution for the entire time range can be easily established when the solutions exist. Under these conditions (59) can be written for the present case but in the time domain as

$$B(\tfrac{\partial}{\partial t})\, p(r,t) = -\frac{4}{\pi} A(\tfrac{\partial}{\partial t})\, I[r, R(t)] \tag{69}$$

where

$$I[r, R(t)] = \int_0^r \frac{d\xi}{\xi(r^2-\xi^2)^{\frac{1}{2}}} \frac{d}{d\xi} \int_\xi^R \frac{d\overline{W}(\rho,s)}{d\rho} \frac{\rho^2 d\rho}{(\rho^2-\xi^2)^{\frac{1}{2}}}, \tag{70}$$

and B and A are differential operators which are similar to Laplace transform functions $\overline{B}(s)$ and $\overline{A}(s)$ respectively. In addition, we can transform (56) by using the same technique. It must be impressed that (69) holds only when $R(t)$ is an increasing function and can be treated as a differential equation for $p(r,t)$ where the right-hand side is known. In this case we get a consistent pressure distribution for the given crack opening. Also, we can consider (69) as a differential equation for $I[r, R(t)]$. Similarly, (60) is modified to

$$B(\tfrac{\partial}{\partial t})\, P(t) = -8A(\tfrac{\partial}{\partial t}) \int_0^{R(t)} r\, I[r, R(t)]\, dr \tag{71}$$

where $P(t)$ is the total load on the crack surface.

For simplicity, if we assume that

$$B\left(\frac{\partial}{\partial t}\right) = b_0 + b_1 \frac{\partial}{\partial t}$$
$$A\left(\frac{\partial}{\partial t}\right) = a_0 + a_1 \frac{\partial}{\partial t} \tag{72}$$

then (69) and (70) are first order differential equations and have solutions as

$$p(r,t) = p_0 e^{-\frac{b_0}{b_1}t} - \frac{4}{\pi b_1} e^{-\frac{b_0}{b_1}t} \int e^{\frac{b_0}{b_1}t} \left(a_0 + a_1 \frac{\partial}{\partial t}\right) I[r,R(t)]\,dt \tag{73}$$

and

$$P(t) = P_0 e^{-\frac{b_0}{b_1}t} - \frac{8}{b_1} e^{-\frac{b_0}{b_1}t} \int e^{\frac{b_0}{b_1}t} \left(a_0 + a_1 \frac{\partial}{\partial t}\right) \int_0^R rI[r,R(t)]\,dr\,dt \tag{74}$$

where p_0 and P_0 are constants of integration and are to be evaluated at $t = 0$. We can also solve (69) and (70) as

$$I[r,R(t)] = M e^{-\frac{a_0}{a_1}t} - \frac{\pi}{4a_1} e^{-\frac{a_0}{a_1}t} \int e^{\frac{a_0}{a_1}t} \left(b_0 + b_1 \frac{\partial}{\partial t}\right) p(r,t)\,dt \tag{75}$$

$$\int_0^{R(t)} rI[r,R(t)]\,dt = N e^{-\frac{a_0}{a_1}t} - \frac{1}{8a_1} e^{-\frac{a_0}{a_1}t} \int e^{\frac{a_0}{a_1}t} \left(b_0 + b_1 \frac{\partial}{\partial t}\right) P(t)\,dt \tag{76}$$

where M and N are constants of integration. However, if $B\left(\frac{\partial}{\partial t}\right)$ and $A\left(\frac{\partial}{\partial t}\right)$ are higher order differential operators, then solutions of (73), (74), (75) and (76) will contain more terms similar to those already obtained, and the computations present no difficulties. It can be shown that the results obtained in this section can be reduced easily to those in an earlier section, when $P(t) = P_0 H(t)$ where $H(t) = 1$ $t > 0$, $H(t) = 0$ $t \leq 0$

Also, similar results can be obtained for isotropic viscoelastic medium by substituting proper values for $\bar{\alpha}(s)$, $\bar{\beta}(s)$, etc. in terms of $\bar{\lambda}(s)$ and $\bar{\mu}(s)$. In this case instead of two functions $\bar{\psi}_k(r, z, s)$, we get only one function since $\bar{\eta}_1(s) = \bar{\eta}_2(s) = 1$. The dependence of spatial coordinates for both isotropic and transverse media is the same, so that many of the conclusions in either cases would be expected to be the same. If the crack is not circular as assumed, the problem becomes difficult but still can be solved. In that case the preliminary information must be expressed in generalized coordinates, and the proper corresponding results will then be derived in the same coordinate system.

Illustrations and Numerical Results

Let us now consider a particular case of a crack on which a uniform pressure is applied. First, we consider the case wherein $P(t)$ = constant. Then

$$p(r,t) = p H(t) \tag{77}$$

where $H(t)$ is the Heavyside unit step function, and

$$\bar{h}_1(r,s) = -\frac{P_0}{\pi s \bar{A}(s)} [1 - R\,\delta(R-r)] \tag{78}$$

On the $z = 0$ plane

$$\bar{U}_z(r,0,s) = -\frac{p_0 \bar{B}(s)}{5\pi \bar{A}(s)} (R^2-r^2)^{\frac{1}{2}} \qquad 0 \le r \le R$$

$$= 0 \qquad R \le r < \infty$$

$$\bar{\sigma}_{zz}(r,0,s) = -\frac{p_0}{s} \qquad 0 \le r \le R$$

$$= -\frac{2p_0}{\pi s}\left[\sin^{-1}\frac{R}{r} - \frac{\frac{R}{r}}{\left[1-\left(\frac{R}{r}\right)^2\right]^{\frac{1}{2}}}\right] \qquad R \le r < \infty \tag{79}$$

Similarly, other quantities can also be computed. Before we can obtain any numerical results we must consider a particular material and its properties. It may be mentioned that the above results are true for any general viscoelastic medium. Now let us consider a medium whose behavior $C_{ijkl}(t)$ is representable by a standard linear solid. The response curve for this model material is given by the following differential equation

$$\frac{\partial \sigma}{\partial t} + \frac{E_1+E_2}{\eta_2}\sigma = E_1\frac{\partial \varepsilon}{\partial t} + \frac{E_1 E_2}{\eta_2}\varepsilon \tag{80}$$

where E_1 and E_2 are elastic constants, η_2 is the viscosity coefficient and τ the relaxation time such that $\eta_2 = E_2\tau$. In general for various $C_{ijkl}(t)$ E's and η's are different in different directions. For a class of high polymers the elastic properties for a transversely isotropic medium are theoretically predicted in [14]. Choosing the values for 125% orientation we get

$$C_{33}^{(e)} = 1.95\, C_{11}^{(e)} \qquad C_{13}^{(e)} = C_{44}^{(e)} = 0.450\, C_{11}^{(e)}$$

$$C_{12}^{(e)} = C_{66}^{(e)} = 0.333\, C_{11}^{(e)} \tag{81}$$

The choice of relaxation times can be based upon the supposition that viscosity does not vary significantly with orientation. Then we get

$$\tau_{11} = \tau \qquad \tau_{33} = 0.513\,\tau$$

$$\tau_{12} = \tau_{66} = 3\,\tau \qquad \tau_{13} = \tau_{44} = 2.221\,\tau \tag{82}$$

Also, it is assumed that in (80) $E_1 = E_2 = E$. This corresponds to having similar springs in the model (Fig 6). It may be mentioned that the values in (81) are presumably close to the actual physical values. Through the use of (81) and (82) various quantities $\bar{\alpha}_i(s), \bar{\beta}_i(s)$, etc. are easily calculated. It is relatively simple to determine large and small time limiting behavior of various quantities in (79). Also the results for $0 < t < \infty$ are obtained using only the first approximation. In Fig 7 the displacement $u_z(r, 0, t)$ is drawn against the dimensionless radius.

Next we consider the same problem for $R = R(t)$. Again using the first approximation as in (72) it can be shown that

$$B\left(\frac{\partial}{\partial t}\right) = 1 + 205.5\frac{\partial}{\partial t}$$
$$A\left(\frac{\partial}{\partial t}\right) = 0.201 + 0.786\frac{\partial}{\partial t} \tag{83}$$

If we assume that $W(r,t)$ in (67) is given as

$$W(r,t) = \Omega(t)\left[R(t)^2 - r^2\right]^{\frac{1}{2}} \tag{84}$$

consistent with the normal loading $p(r,t) = p_0 H(t)$ where $\Omega(t)$ and $R(t)$ are unknown functions of time. Also, $\Omega(t) = \Omega_0$ at $t = 0$; then

$$\Omega_o = -\frac{2P_o}{\pi} \Big/ \left[\frac{\bar{A}(s)}{s\bar{B}(s)}\right]_{s=\infty} = \frac{2P_o}{0.201\,\pi}$$

Substitute (84) into (75) we get

$$\frac{\Omega(t)}{\Omega_o} = 30.4 - 29.4\, e^{-0.256\frac{t}{\tau}} \tag{85}$$

Now put (84) into (76) and solve for $R(t)$ then

$$\frac{R(t)}{R_o} = e^{0.0482\frac{t}{\tau}} \left(1.545 - 0.545\, e^{-0.256\frac{t}{\tau}}\right)^{0.310} \tag{86}$$

where R_o is the initial radius of the crack. In order to obtain reasonable results, we must assume $R_o \neq 0$. Some of these results are shown in Figures 8 and 9. It can be seen from the curve in Fig 8 that $\Omega(t)$ assumes a finite limit for $t=0$ and $t=\infty$. But $R(t)$ in Fig 9 becomes infinite as can be expected from the physical reasoning. It is interesting to note that the crack must propagate away from its center, since the total load is increasing with time. Another interesting observation is that $R(t)$ varies approximately linearly with time. This suggests a relation

$$\frac{dR(t)}{dt} \cong V \tag{87}$$

where V is the slope of the curve, and is nearly constant. In many earlier investigations a constant rate of crack propagation has often been assumed. Furthermore, there is no difficulty in obtaining analytical results for cracks of different shapes.

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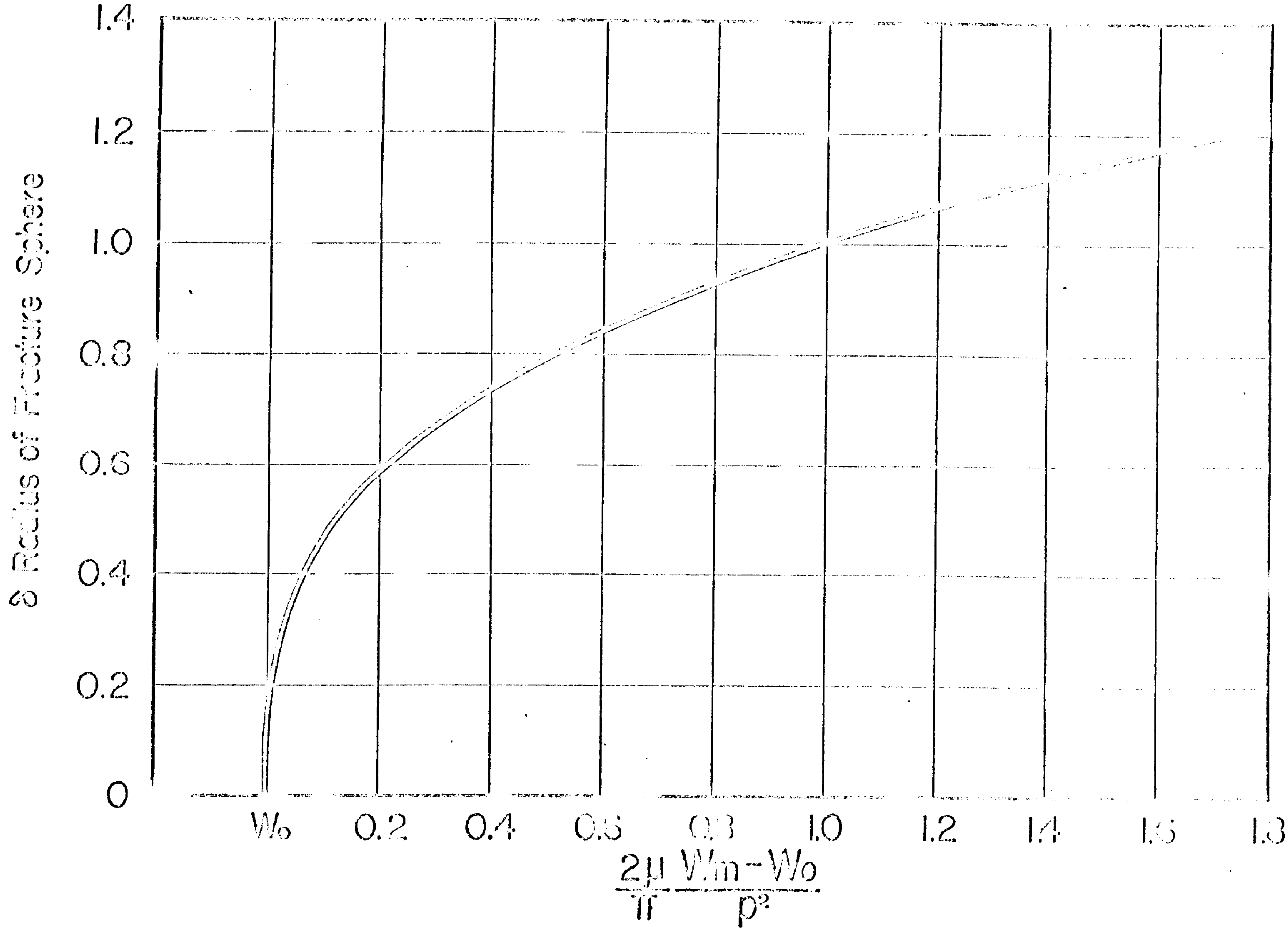


Fig. 1 Reflected Strain Energy vs Size of Fracture

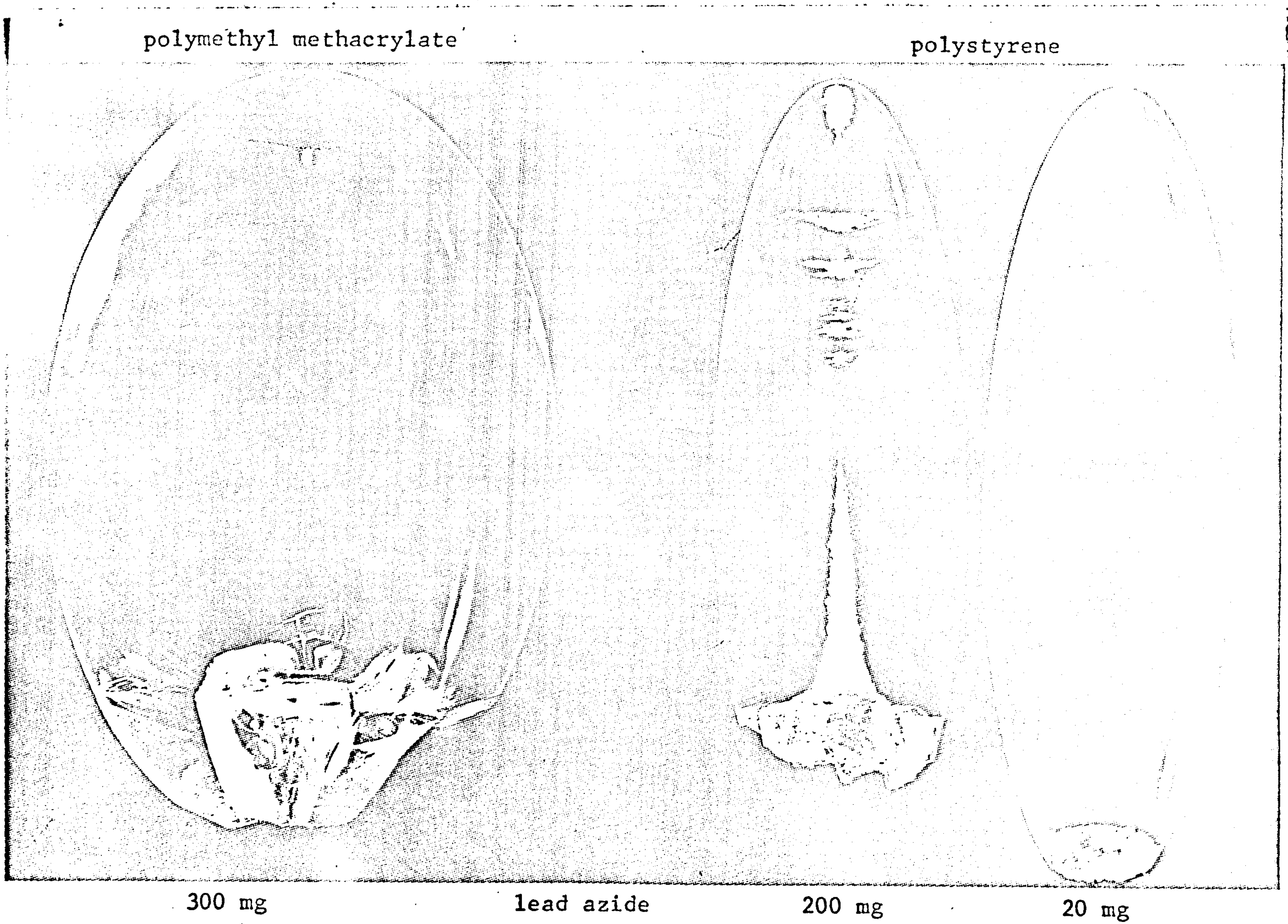
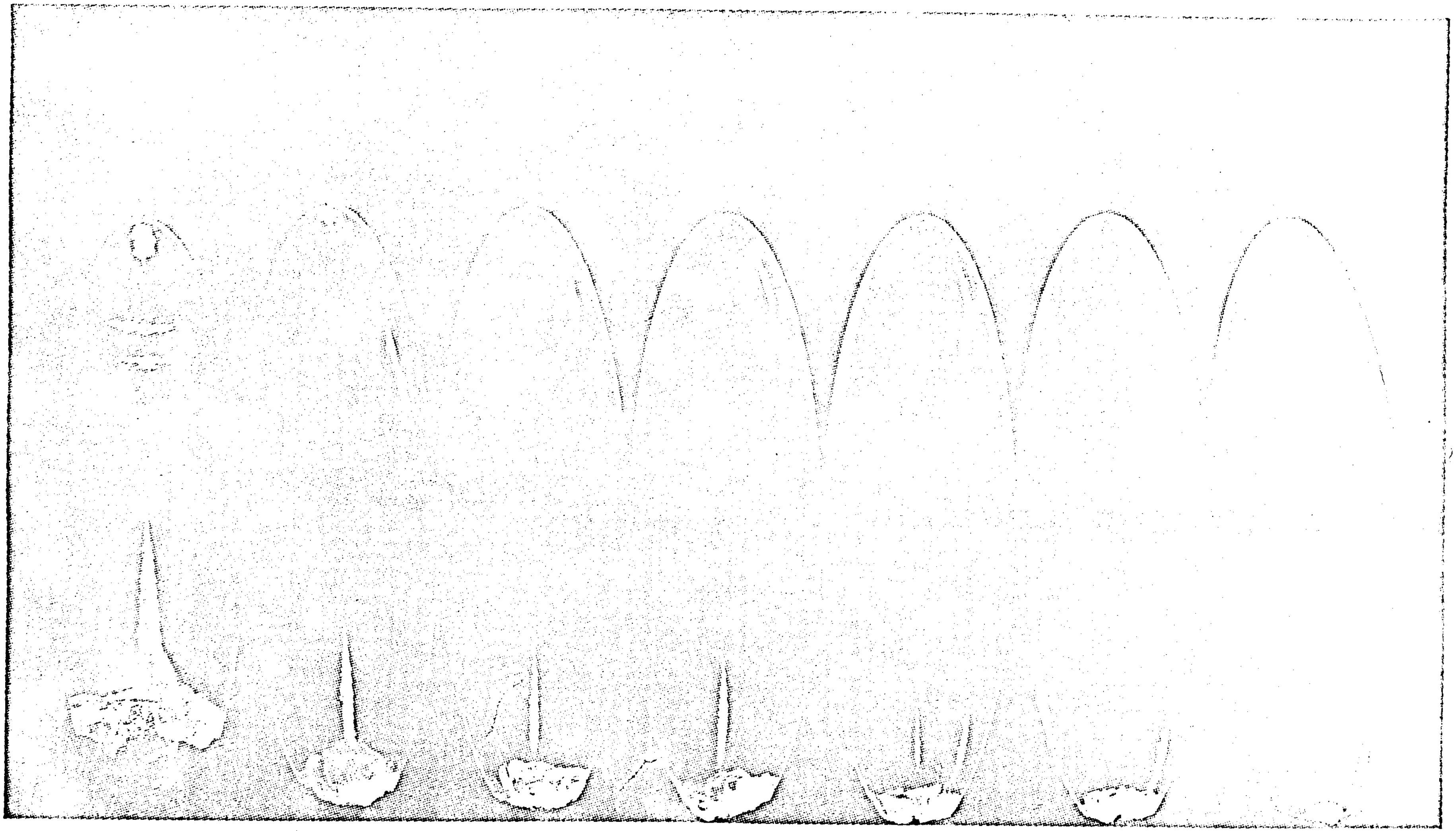


Fig. 2 Internal Fracture of Polymeric Solids



200 mg 100 mg 70 mg 30 mg 25 mg 20 mg 15 mg

lead azide used

Fig. 3 Internal Fracture of Polystyrene

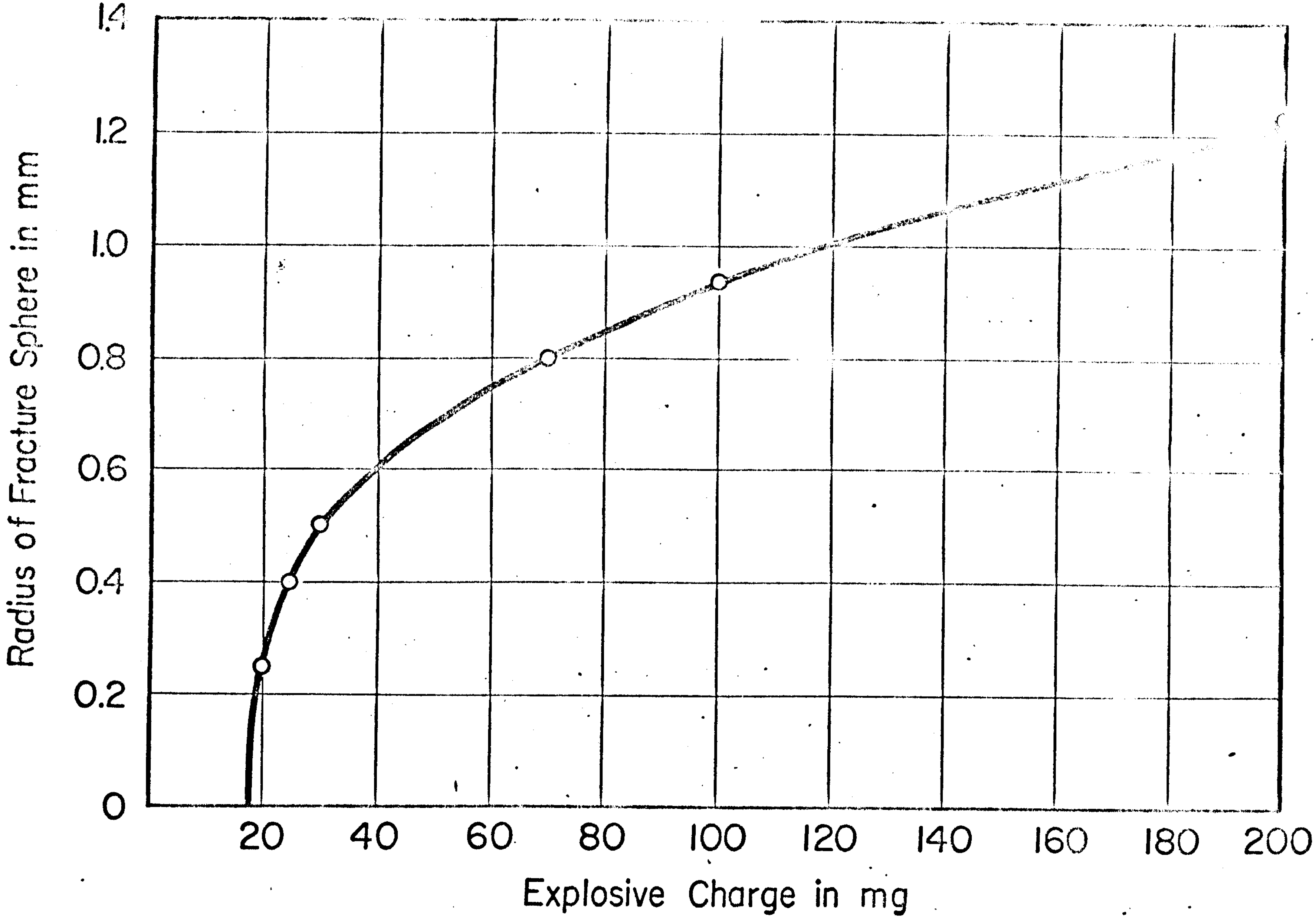


Fig. 4 Explosive Charge vs Fracture Size

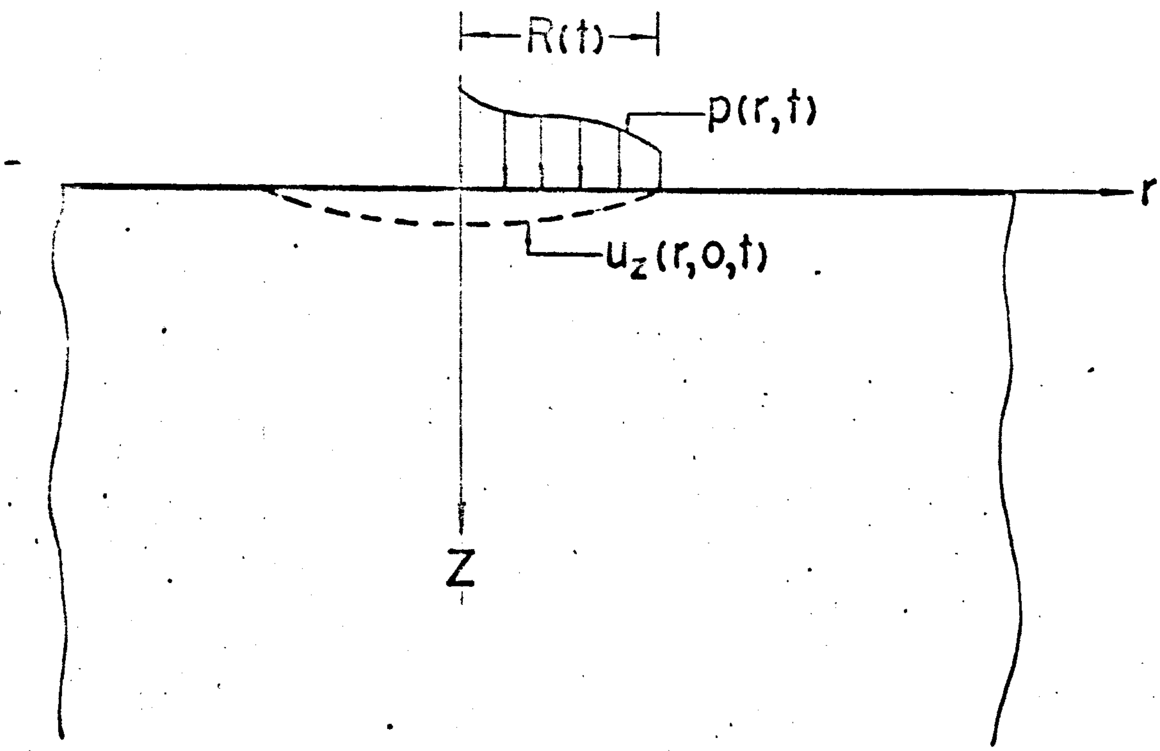
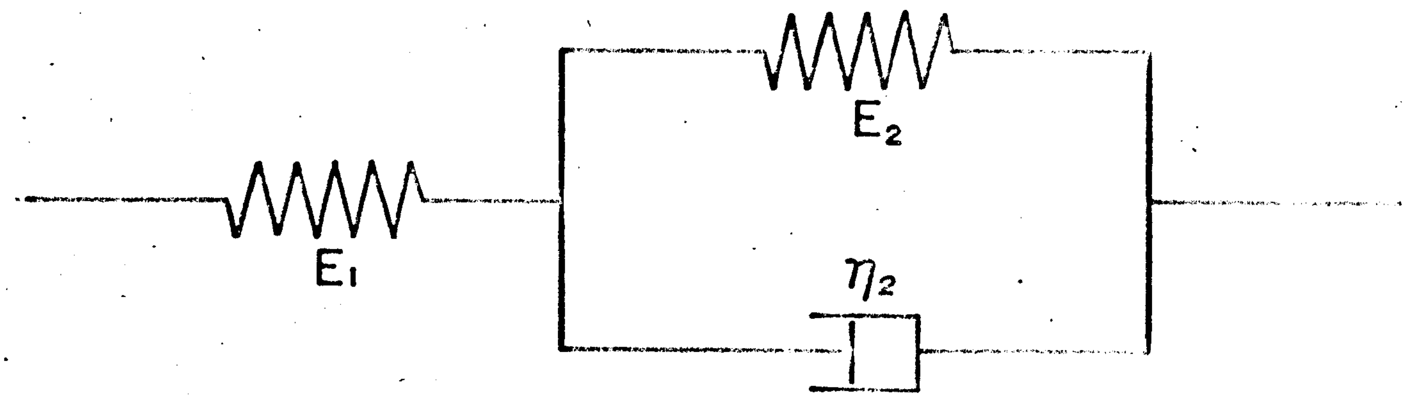


Fig. 5   Schematic Diagram of a Crack



$$\frac{d\sigma}{dt} + \frac{E_1+E_2}{\eta_2}\sigma = E_1\frac{d\epsilon}{dt} + \frac{E_1E_2}{\eta_2}\epsilon\ ; \quad \eta_2 = E_2\tau$$

Fig. 6 Schematic Diagram of a Standard Linear Viscoelastic Medium

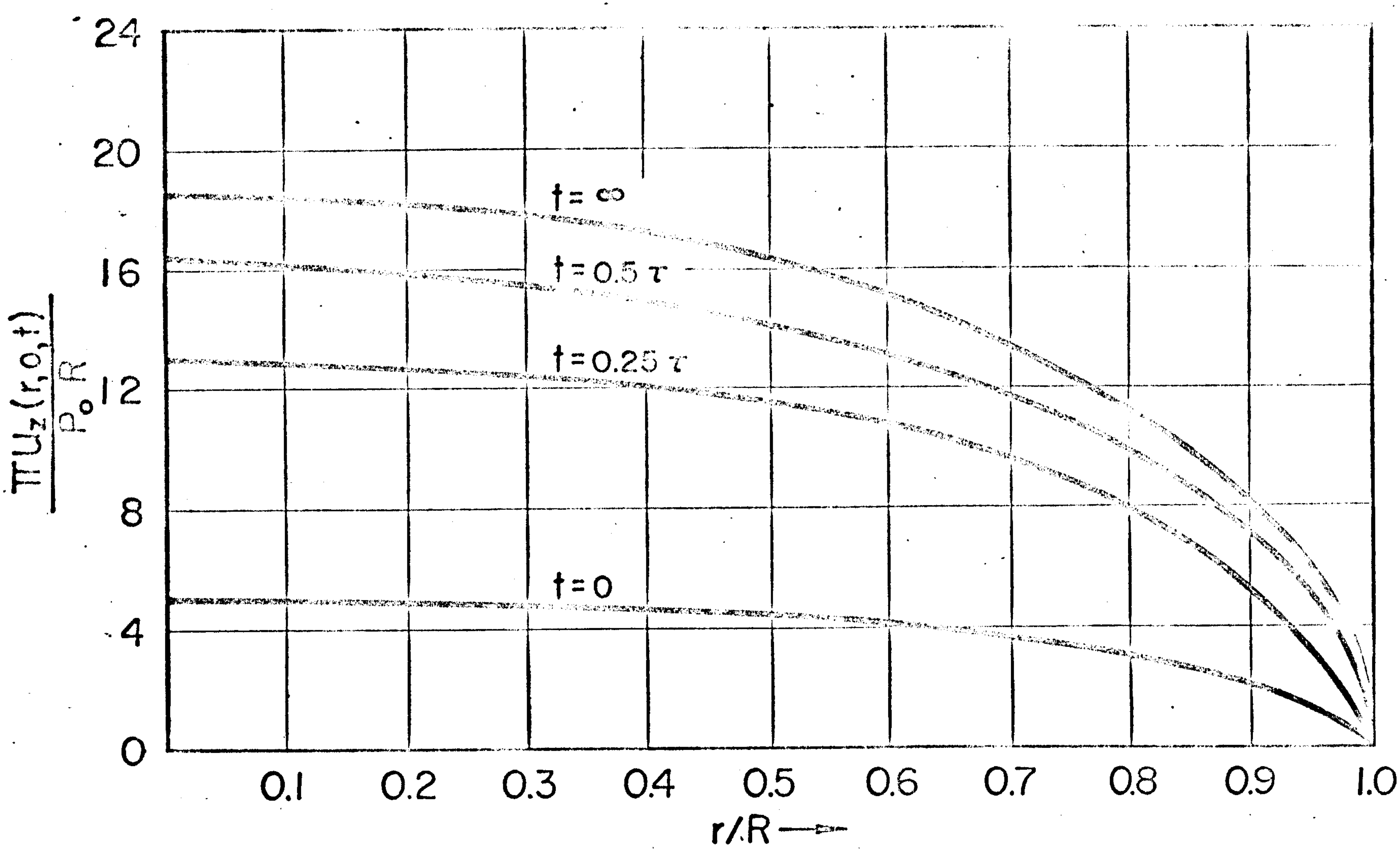


Fig. 7 Crack Opening Under Constant Uniform Loading

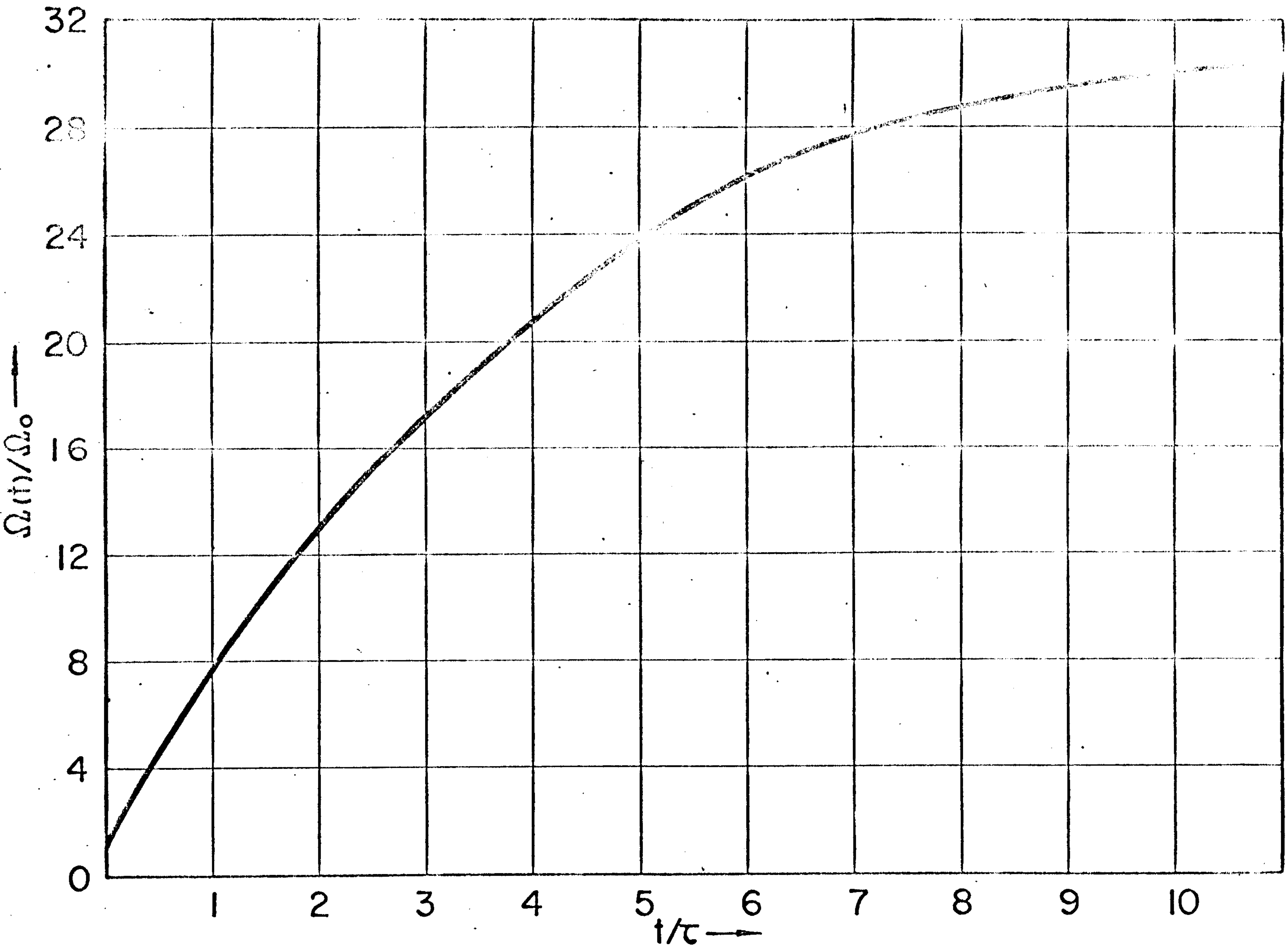


Fig. 8 Variation of $\Omega(t)$ in $U_Z(r,t) = \Omega(t)\left[R^2(t)-r^2\right]^{\frac{1}{2}}$ Under Uniform Loading

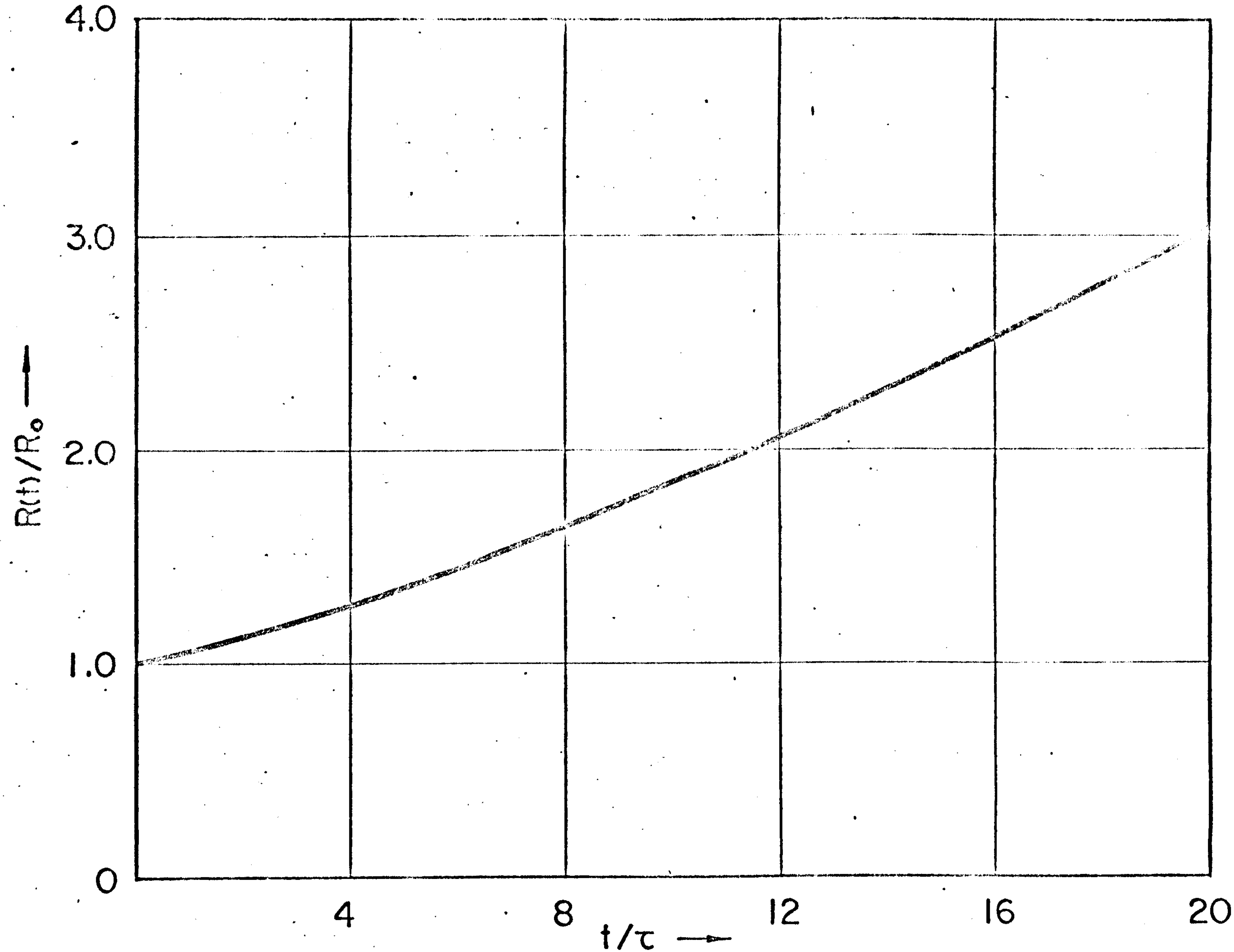


Fig. 9 Propagation of Crack Under Constant Uniform Loading